Effective models for non-perfectly conducting thin coaxial cables

Geoffrey Beck, Sebastien Imperiale, Patrick Joly

Effective models for non-perfectly conducting thin coaxial cables

Geoffrey Beck[1,*], **Sébastien Imperiale**[2], **Patrick Joly**[1]
[1]POEMS (CNRS-INRIA-ENSTA Paristech), Palaiseau, France
[2]Inria & LMS, Ecole polytechnique, CNRS, Université Paris-Saclay, France
*Email: geoffrey.beck.poems@gmail.com

Abstract

Continuing past work on the modelling of coaxial cables, we investigate the question of the modeling of non-perfectly conducting thin coaxial cables. Starting from 3D Maxwell's equations, we derive, by asymptotic analysis with respect to the (small) transverse dimension of the cable, a simplified effective 1D model. This model involves a fractional time derivatives that accounts for the so-called skin effects in highly conducting regions.

Keywords: Maxwell's equations, Coaxial cables, Asymptotic analysis

Statement of the problem

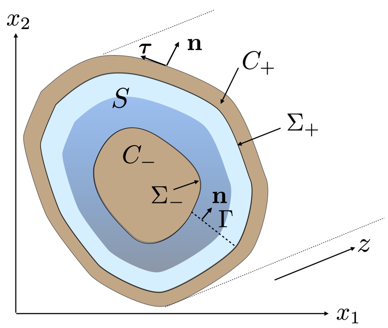


Figure 1: Section the coaxial cable. Σ_+ and Σ_- are the outer and inner boundary of S.

Denoting $\delta > 0$ a small parameter, we consider a family of (thin) domains $\Omega^\delta = \mathcal{G}_\delta(\Omega)$ where

$$\mathcal{G}_\delta : (x_1, x_2, z) \longrightarrow (\delta x_1, \delta x_2, z).$$

and Ω is the disjoint union of a conducting domain Ω_c and a dielectric one Ω_d,

$$\Omega_c = C \times \mathbb{R}, \quad \Omega_d = S \times \mathbb{R},$$

where $C = C^+ \cup C^-$, C^+ corresponding to the outer metallic shield and C^- to the inner metallic wire and S is non-simply connected, see Figure 1. Accordingly, we have, with obvious notation

$$\Omega^\delta = \Omega^\delta_d \cup \Omega^\delta_c.$$

We are interested in the solution (E^δ, H^δ) of 3D Maxwell's equation in Ω^δ :

$$\begin{cases} \varepsilon^\delta\, \partial_t E^\delta + \sigma^\delta\, E^\delta - \mathbf{curl}\ H^\delta = \mathbf{j}^\delta, \\ \mu^\delta\, \partial_t H^\delta + \mathbf{curl}\ E^\delta = \mathbf{0}, \end{cases} \tag{1}$$

with zero initial data. In Ω^δ_c, $(\varepsilon^\delta, \mu^\delta)$ are constant equal to (ε_c, μ_c) and $\mathbf{j}^\delta = 0$. In Ω^δ_d, $(\varepsilon^\delta, \mu^\delta)$ do not depend on z and are obtained by a scaling in the transverse variable $\mathbf{x}_T = (x_1, x_2)$ of fixed distributions in the reference domain Ω_d, for instance

$$\varepsilon^\delta(\mathbf{x}_T, z) = \varepsilon(\mathbf{x}_T/\delta).$$

The source term $\mathbf{j}^\delta$ is defined similarly, moreover it is compactly supported, it has no longitudinal component and is divergence free. The conductivity is weak in the dielectric Ω^δ_d, but very high in Ω^δ_c. More precisely

$$\sigma^\delta(\mathbf{x}_T, z) = \begin{cases} \delta^{-4}\sigma_c & \text{in } \Omega^\delta_c, \\ \delta\, \sigma(\mathbf{x}_T/\delta), & \text{in } \Omega^\delta_d. \end{cases} \tag{2}$$

Note that the $O(\delta^{-4})$ magnitude of σ^δ in Ω^δ_c gives rise to a skin depth in $O(\delta^2)$, small with respect to δ.

Our approach consists in obtaining the formal behaviour of the solution for small δ. To do so, we propose two distinct assymptotic expansions of the solution Ω^δ_d and Ω^δ_c that we match using transmission conditions. We present below our main results.

Electromagnetic field in the dielectric

We introduce the following notations.

- ∇ *for the 2D transverse gradient in* $\mathbf{x}_T$, *identified to a 3D vector with third component 0,*
- $S_\Gamma := S \setminus \Gamma$ *where* Γ *is a cut that makes* S_Γ *simply connected (see Figure 1),*
- $[\cdot]_\Gamma$ *for the jump across* Γ *in the direction* $\boldsymbol{n}$,
- $\widetilde{\nabla}$ *is the 2D transverse gradient in* S_Γ,
- $\partial_{\boldsymbol{n}}$ *is the normal derivative, and* $\partial_{\boldsymbol{\tau}}\psi = \widetilde{\nabla}\psi \cdot \boldsymbol{\tau}$ *the tangential derivative.*

We obtain that, for small δ and all $\mathbf{x}_T \in S^\delta$,

$$
\begin{aligned}
E^\delta(\mathbf{x}_T, z, t) \sim\ & V^\delta(z,t)\, \nabla\varphi_e(\mathbf{x}_T/\delta) \\
& + \delta \left(\textstyle\int_0^t V^\delta(z,s)\, ds\right) \nabla\varphi_r(\mathbf{x}_T/\delta), \\
& + \delta\, \partial_z V^\delta(z,t)\, (\varphi_e - \varphi_m)(\mathbf{x}_T/\delta)\, \mathbf{e}_z,
\end{aligned}
$$

$$
\begin{aligned}
H^\delta(\mathbf{x}_T, z, t) \sim\ & I^\delta(z,t)\, \nabla\psi_m(\mathbf{x}_T/\delta), \\
& + \delta \left(\textstyle\int_0^t \partial_t^{\frac{1}{2}} I^\delta(z,s)\, ds\right) \nabla\psi_r(\mathbf{x}_T/\delta), \\
& + \delta\, \partial_z I^\delta(z,t)\, (\psi_e - \psi_m)(\mathbf{x}_T/\delta)\, \mathbf{e}_z,
\end{aligned}
$$

where $\mathbf{e}_z = (0,0,1)^t$. Moreover:

i) The potential $\varphi_e \in H^1(S)$ satisfies,

$$\operatorname{div} \varepsilon \nabla \varphi_e = 0 \ (S), \ \ \varphi_e = 0 \ (\Sigma_+), \ \ \varphi_e = 1 \ (\Sigma_-),$$

and the same for φ_m with μ^{-1} instead of ε.

ii) The potential $\psi_m \in H^1(S_\Gamma)$ satisfies

$$\operatorname{div} \mu \nabla \psi_m = 0 \ (S_\Gamma), \ \ \partial_{\boldsymbol{n}} \psi_m = 0 \ (\partial S),$$

and $[\psi_m]_\Gamma = 1$, $[\partial_{\boldsymbol{n}} \psi_m]_\Gamma = 0$. The same holds for ψ_e with ε^{-1} instead of μ. Moreover

$$\int_S \mu\, \psi_e = \int_S \mu\, \psi_m = 0.$$

iii) The function $\varphi_r \in H_0^1(S)$ is the solution of

$$\operatorname{div} \varepsilon \nabla \varphi_r = -\operatorname{div} \sigma \nabla \varphi_e.$$

iv) The function $\psi_r \in H^1(S)$ satisfies

$$\operatorname{div} \mu \nabla \psi_r = 0 \ (S), \ \ \mu\, \partial_{\boldsymbol{n}} \psi_r = -\sqrt{\frac{\mu_c}{\sigma_c}}\, \partial_{\boldsymbol{\tau}}^2 \psi_m \ (\partial S).$$

v) The electric potential $V^\delta(z,t)$ and current $I^\delta(z,t)$ are 1D unknowns governed by generalized telegrapher's equations:

$$
\left\{
\begin{aligned}
& C\, \partial_t V^\delta + \delta\, G\, V^\delta + \partial_z I^\delta = j, \\
& L\, \partial_t I^\delta + \delta\, R\, \partial_t^{\frac{1}{2}} I^\delta + \partial_z V^\delta = 0,
\end{aligned}
\right. \tag{3}
$$

where $j(z,t)$ is an effective source term,

$$j(z,t) = \int_S \mathbf{j}(\mathbf{x}_T, z, t) \cdot \nabla\varphi_e(\mathbf{x}_T), \tag{4}$$

and $\partial_t^{\frac{1}{2}}$ is the square root derivative in the sense of Caputo

$$\partial_t^{\frac{1}{2}} u(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial_\tau u(\tau)}{\sqrt{t-\tau}}\, d\tau.$$

As in [1], the capacity C, inductance L and conductance G are given by:

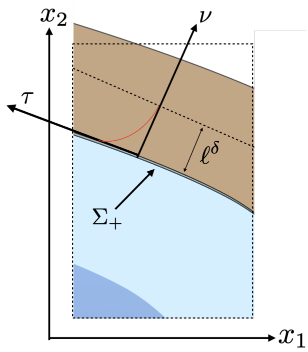
$$C = \int_S \varepsilon \left|\nabla\varphi_e\right|^2, \ L = \int_S \mu \left|\widetilde{\nabla}\psi_m\right|^2, \ G = \int_S \sigma \left|\nabla\varphi_e\right|^2.$$

Moreover, we obtain an explicit expression for the resistance R, which takes into account skin effects:

$$R = \int_{\partial S} \sqrt{\frac{\mu_c}{\sigma_c}} \left|\partial_{\boldsymbol{\tau}} \psi_m\right|^2. \tag{5}$$

This generalizes formulas of the literature (see [2], chapter 13) already derived in very simple cases.

Electric field in the outer conductor



In the rescaled conducting domain C_+ the electromagnetic fields are described using tangential and normal coordinates (τ, ν). The penetration depth ℓ^δ of the fields is in $O(\delta)$.

L^+ being the length of Σ_+, one shows that there exists a 3D field

$$E^+ : [0, L^+] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^3$$

such that in $C_+^\delta \times \mathbb{R}$ and for small δ,

$$E^\delta(\mathbf{x}_T(\tau,\nu), z, t) \sim \delta^2 E^+(\tau/\delta, \nu/\delta^2, z, t),$$

(and a similar property holds for the magnetic field). The important fact is that the component E_z^+ is solution of the 1D heat equation

$$\mu_c \sigma_c\, \partial_t E_z^+ - \partial_\nu^2 E_z^+ = 0, \tag{6}$$

and thus satisfies, at the boundary $\nu = 0$:

$$\partial_\nu E_z^+ + \sqrt{\mu_c \sigma_c}\, \partial_t^{\frac{1}{2}} E_z^+ = 0.$$

The above equation is used when writing transmission conditions across $\Sigma_+^\delta \times \mathbb{R}$.

This explains the appearance of $\partial_t^{\frac{1}{2}}$ in the effective model (3).

References

[1] G. Beck, S. Imperiale, P. Joly *Mathematical modelling of multi conductor cables*, Disc. & Cont. Dynamical Systems, (2014).

[2] W. H. Hayt, J. A. Buck, *Engineering Electromagnetics*, 6th Edition, (2001).